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Gert-Jan C. Lokhorst & Lou Goble

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Gert-Jan C. Lokhorst

Faculty of Philosophy,
Erasmus University Rotterdam,
PO Box 1738,
3000 DR Rotterdam,
The Netherlands

email: lokhorst@fwb.eur.nl

Lou Goble

Department of Philosophy,
Willamette University,
900 State Street,
Salem, OR 97301,
USA

email: lgoble@willamette.edu

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G.J.C. Lokhorst, June 25, 2005,
new e-mail address: `g.j.c.lokhorst@tbm.tudelft.nl`

Mally's Deontic Logic

Abstract In 1926, Mally presented the first formal system of deontic logic. His system had several consequences which Mally regarded as surprising but defensible. It also, however, has the consequence that A is obligatory if and only if A is the case, which is unacceptable from the point of view of any reasonable deontic logic. We describe Mally's system and discuss how it might reasonably be repaired.

1. Introduction

In 1926, the Austrian philosopher Ernst Mally (1879–1944) proposed the first formal system of deontic logic. In the book in which he presented this system, *The Basic Laws of Ought: Elements of the Logic of Willing*, he gave the following motivation for his enterprise:

> In 1919, everybody was using the word self-determination. I wanted to obtain a clear understanding of this word. But then, of course, I immediately stumbled on the difficulties and obscurities surrounding the concept of Ought, and the problem changed. The concept of Ought is the basic concept of the whole of ethics. It can only serve as a usable foundation for ethics when it is captured in a system of axioms. In the following I will present such an axiomatic system. (*Im Jahre 1919 wurde mir das Wort Selbstbestimmung, das in aller Leute Munde war, Anlaß eines Versuches, mir einen klaren Begriff zu dem Wort zu bilden. Natürlich stieß ich dabei alsbald auf die Schwierigkeiten und Dunkelheiten des Sollensbegriffes: das Problem wandelte sich. Grundbegriff aller Ethik, kann der Begriff des Sollens ein brauchbares Fundament ihres Aufbaus nur geben, wenn er in einem System von Axiomen festgelegt ist. Ein solches Axiomensystem führe ich hier vor*—Mally 1926, Preface, p. I.)

As Mally's words indicate, he was not primarily interested in deontic logic for its own sake; he mainly wanted to use it to lay the foundation of "an exact system of pure ethics" (*eine exakte reine Ethik*). More than half of his book is devoted to the development of this exact system of ethics. In the following, we will, however, concentrate on the formal part of his work, both because this is its "hard core" and because it is the part that has attracted the most interest.

2. Mally's Formal Language

Mally based his formal system on the classical propositional calculus (**PC**) as formulated in Whitehead's and Russell's *Principia Mathematica* (vol. 1, 1910).

The non-deontic part of Mally's system had the following vocabulary: the sentential letters A, B, C, P and Q (these symbols refer to states of affairs), the sentential variables M and N, the sentential constants V (the *Verum*, Truth) and Λ (the *Falsum*, Falsity), the propositional quantifiers $\exists$ and $\forall$, and the usual connectives, for which we use a more modern notation, $\neg$, &, $\vee$, $\rightarrow$ and $\leftrightarrow$. Mally read $A \rightarrow B$ as "A implies B" and as "If A, then B", and in keeping with classical logic, he considered this equivalent to $\neg(A \,\&\, \neg B)$. Similarly, V is such that $A \rightarrow V$ is provable for every formula A. Λ is defined by $\Lambda = \neg V$.

The deontic part of Mally's vocabulary consisted of the unary connective !, the binary connectives f and ∞, and the sentential constants U and $\mho$. (Today in deontic logic it would be most common to use O in place of Mally's !; we shall follow his practice, however, in order to keep the deontic part of his system as close to the original as possible, and to avoid importing contemporary assumptions into the formalism.)

Mally read this vocabulary as follows:

- $!A$ as "A ought to be the case" (A *soll sein*) or as "let A be the case" (*es sei* A).
- $A \; f \; B$ as "A requires (calls for) B" (A *fordert* B).
- $A \infty B$ as "A and B require each other."
- U as "the unconditionally obligatory" (*das unbedingt Geforderte*).
- $\mho$ as "the unconditionally forbidden" (*das unbedingt Verbotene*).

Furthermore, he defined f, ∞ and $\mho$ by:

Def. f. $A \; f \; B = A \rightarrow !B$ (Mally 1926, p. 12)
Def. ∞. $A \infty B = (A \; f \; B) \,\&\, (B \; f \; A)$
Def. $\mho$. $\mho = \neg U$

In addition to reading $!A$ as "it ought to be the case that A" in the sense that A is obligatory or that A is normatively required, Mally also read $!A$ as "A is desirable" or "I want it to be the case that A". For example, a person might say "I ought to be rich and famous" or "It ought to be the case that I am rich and famous" to indicate that she wants to be rich and famous. With this reading Mally's formal system was as much a theory about *Wollen* (willing) as a theory about *Sollen* (ought to be the case). This explains the subtitle of his book. In modern deontic logic, the basic deontic connective O is seldom taken in this way, and we shall disregard this broader interpretation. It does not affect the results to follow.

There are some other noteworthy differences between Mally's deontic logic and contemporary systems:

- Mally was only interested in the deontic status of states of affairs; he paid no special attention to the deontic status of actions. Thus, his *Deontik* was a theory about *Seinsollen* (what ought to be the case) rather than

Tunsollen (what ought to be done). Many deontic logicians nowadays regard the concept of *Tunsollen* as fundamental. (For grammatical convenience, we often use the term "obligatory" here as equivalent to this "ought-to-be", Mally's "Seinsollen".)

- In modern deontic logic, the notions of prohibition F, permission P and waiver W are usually defined in terms of obligation O: $FA = O\neg A$, $PA = \neg FA$, $WA = \neg OA$. Such concepts are not to be found in Mally's book (though they could, of course, have been included by the same definitions).
- Related to this, the principle that whatever ought to be is permitted, $!A \rightarrow \neg!\neg A$, or $!A \rightarrow PA$, which is the hallmark of much modern deontic logic, is nowhere mentioned by Mally, although it is derivable in his system.

3. Mally's Axioms

Mally adopted the following informal deontic principles as the foundation for his system (Mally 1926, pp. 15–19):

(i) If A requires B and B implies C, then A requires C.

(ii) If A requires B and if A requires C, then A requires B and C.

(iii) The conditional requirement "A requires B" is equivalent to the unconditional "It ought to be that A implies B (if A then B)".

(iv) There is an "unconditionally obligatory" state of affairs, i.e., there is a state of affairs, U, such that it ought to be that U.

(v) The unconditionally obligatory does not require its own negation.

Mally did not offer much support for these principles. They simply seemed intuitively plausible to him. He then formalized them with the axioms (Mally 1926, pp. 15–19):

$$\begin{array}{ll} \text{I.} & ((A \mathrel{f} B) \,\&\, (B \rightarrow C)) \rightarrow (A \mathrel{f} C) \\ \text{II.} & ((A \mathrel{f} B) \,\&\, (A \mathrel{f} C)) \rightarrow (A \mathrel{f} (B \,\&\, C)) \\ \text{III.} & (A \mathrel{f} B) \leftrightarrow !(A \rightarrow B) \\ \text{IV.} & \exists U !U \\ \text{V.} & \neg(U \mathrel{f} \mho) \end{array}$$

which would be added to **PC**, the classical propositional calculus.

Axiom IV is strange. As formulated, it reflects the reading of the informal principle (iv) well enough, but at the same time it reveals some confusion about the grammatical status of the symbol U. Throughout the text U seems to be treated as a propositional constant, analogous to V. In that case, however, it should not occur inside a quantifier, and one should write $!U$ instead. This seems the most natural representation of what Mally has in mind. If, however,

U were to be genuinely construed as a variable, then Axiom IV is redundant. $!A \to !A$ is a tautology, so we have $!A \textsf{ f } A$ by Def. $\textsf{f}$, whence $!(!A \to A)$ by Axiom III→, whence $\exists M!M$ by existential generalization. Axiom IV on this construal seems to add nothing to this. Furthermore, Axiom IV is the only axiom or theorem mentioned by Mally in which U occurs as an explicitly bound variable. In Axiom V and in theorems (15)–(17), (20)–(21), (23), (23′) and (27)–(35) (to be displayed below), U is either a constant or, perhaps, a free variable (ranging over unconditionally obligatory states of affairs). It should be treated in the same way in the formalization of (iv).

For these reasons, we replace Axiom IV, as stated by Mally, by

$$\text{IV.} \quad !U$$

Mally, himself, could hardly have objected to this version of Axiom IV because it is equivalent with his theorem (23′), i.e., $V \textsf{ f } U$, in virtue of Def. $\textsf{f}$. In the following "Axiom IV" will always refer to our version of Axiom IV rather than Mally's. (Menger (1939, p. 57) and Føllesdal and Hilpinen (1981, pp. 2–3) made the same amendment.)

Using Def. $\textsf{f}$, Axioms I–V may also be written as follows (Mally 1926, pp. 15–19 and p. 24):

I′.	$((A \to !B) \,\&\, (B \to C)) \to (A \to !C)$
II′.	$((A \to !B) \,\&\, (A \to !C)) \to (A \to !(B \,\&\, C))$
III′.	$(A \to !B) \leftrightarrow !(A \to B)$
IV′.	$V \textsf{ f } U$ (or $V \to !U$)
V′.	$\neg(U \to !\mathit{\Omega})$

4. Mally's Theorems

Given his axioms I–V, Mally derived the following thirty-six theorems (Mally 1926, pp. 20–34). This list follows Mally's numeration. (Note that there are two entries at (23).)

(1)	$(A \textsf{ f } B) \to (A \textsf{ f } V)$
(2)	$(A \textsf{ f } \mathit{\Lambda}) \leftrightarrow \forall M(A \textsf{ f } M)$
(3)	$((M \textsf{ f } A) \vee (M \textsf{ f } B)) \to (M \textsf{ f } (A \vee B))$
(4)	$((M \textsf{ f } A) \,\&\, (N \textsf{ f } B)) \to ((M \,\&\, N) \textsf{ f } (A \,\&\, B))$
(5)	$!P \leftrightarrow \forall M(M \textsf{ f } P)$
(6)	$(!P \,\&\, (P \to Q)) \to !Q$
(7)	$!P \to !V$
(8)	$((A \textsf{ f } B) \,\&\, (B \textsf{ f } C)) \to (A \textsf{ f } C)$
(9)	$(!P \,\&\, (P \textsf{ f } Q)) \to !Q$
(10)	$(!A \,\&\, !B) \leftrightarrow !(A \,\&\, B)$
(11)	$(A \infty B) \leftrightarrow !(A \leftrightarrow B)$
(12)	$(A \textsf{ f } B) \leftrightarrow (A \to !B) \leftrightarrow !(A \to B) \leftrightarrow !\neg(A \,\&\, \neg B) \leftrightarrow !(\neg A \vee B)$

(13) $(A \to !B) \leftrightarrow \neg(A \,\&\, \neg !B) \leftrightarrow (\neg A \vee !B)$
(14) $(A \mathrel{f} B) \leftrightarrow (\neg B \mathrel{f} \neg A)$
(15) $\forall M (M \mathrel{f} U)$
(16) $(U \to A) \to !A$
(17) $(U \mathrel{f} A) \to !A$
(18) $!!A \to !A$
(19) $!!A \leftrightarrow !A$
(20) $(U \mathrel{f} A) \leftrightarrow (A \infty U)$
(21) $!A \leftrightarrow (A \infty U)$
(22) $!V$
(23) $V \infty U$
(23′) $V \mathrel{f} U$
(24) $A \mathrel{f} A$
(25) $(A \to B) \to (A \mathrel{f} B)$
(26) $(A \leftrightarrow B) \to (A \infty B)$
(27) $\forall M (\mho \mathrel{f} \neg M)$
(28) $\mho \mathrel{f} \mho$
(29) $\mho \mathrel{f} U$
(30) $\mho \mathrel{f} \Lambda$
(31) $\mho \infty \Lambda$
(32) $\neg(U \mathrel{f} \Lambda)$
(33) $\neg(U \to \Lambda)$
(34) $U \leftrightarrow V$
(35) $\mho \leftrightarrow \Lambda$

5. Surprising Consequences

Many of these theorems seem natural and are common in deontic logic. Some, however, are likely to raise questions. Mally himself considered theorems (1), (2), (7), (22) and (27)–(35) "surprising" (*befremdlich*) or even "paradoxical" (*paradox*). He viewed (34) and (35) as the most surprising of his surprising theorems. But Mally's reasons for calling these theorems surprising are puzzling if not confused.

Consider, for example, theorem (1). Mally interpreted this theorem as follows: "if A requires B, then A requires everything that is the case" (Mally 1926, p. 20). He regarded this as a surprising claim, and we agree. However, Mally's interpretation of (1) is not warranted. (1) only says that if A requires B, then A requires the *Verum.* The expression "if A requires B, then A requires everything that is the case" is to be formalized as follows:

$$(1') \quad (A \mathrel{f} B) \to (C \to (A \mathrel{f} C))$$

This formula is an immediate consequence of (1) in virtue of Axiom I. In other words, Mally should have reasoned as follows: (1′) is surprising; but (1′) is an immediate consequence of (1) in virtue of Axiom I; Axiom 1 is uncontroversial; so (1) is to be regarded as surprising.

A similar pattern is to be seen in many of Mally's other remarks about theorems which surprised him. He generally read too much into them and confused them with some of the consequences they had in his system:

- Mally was surprised by (2) because he thought that it says that if A requires B and B is not the case, then A requires every state of affairs whatsoever (Mally 1926, p. 21). But (2) says no such thing. Mally's paraphrase is a paraphrase of $(A \mathsf{f} B) \to (\neg B \to (A \mathsf{f} C))$ (a consequence of (2) in virtue of Axiom I) rather than (2).

- Mally paraphrased (7) as "if anything is required, then everything that is the case is required" (Mally 1926, p. 28), which may indeed be regarded as surprising. However, Mally's paraphrase corresponds with $!A \to (B \to !B)$ (a consequence of (7) in virtue of Axiom I) rather than (7).

- Mally paraphrased (22) as "the facts ought to be the case" (Mally 1926, p. 24). We grant that this is a surprising claim. But the corresponding formula in Mally's language is $A \to !A$ (a consequence of (22) in virtue of Axiom I), not (22).

- Mally read (27) as "if something which ought not to be the case is the case, then anything whatsoever ought to be the case" (Mally 1926, pp. 24, 33), but this is a paraphrase of $!\neg A \to (A \to !B)$ (a theorem of Mally's system) rather than (27).

- Mally paraphrased (33) as "what is not the case is not obligatory" (Mally 1926, p. 25) and as "everything that is obligatory is the case" (Mally 1926, p. 34). These assertions are indeed surprising, but Mally's readings of (33) are not warranted. They are paraphrases of $!A \to A$ rather than (33).

- Mally made the following remark about (34) and (35):

  > The latter sentences, which seem to identify being obligatory with being the case, are surely the most surprising of our "surprising consequences." (*Diese letzten Sätze, die Seinsollen und Tatsächlichsein zu identifizieren scheinen, sind unter unseren "befremdlichen Folgerungen" wohl die befremdlichsten*—Mally 1926, p. 25.)

  However, (34) and (35) do not assert that being obligatory is equivalent with being the case, for the latter statement should be formalized as $A \leftrightarrow !A$. The latter formula is a theorem of Mally's system, as will be shown in a moment, but it is not to be found in Mally's book.

Mally regarded theorems (28)–(32) as surprising because of their relationships with certain other surprising theorems:

- (28)–(30) are instantiations of (27). But this is not sufficient to call these theorems surprising. Mally actually viewed (28) as less surprising than (27): one might use it to justify retaliation and revenge (Mally 1926, p. 24).
- (31) implies (28)–(30) and is therefore at least as surprising as those theorems.
- Mally viewed (32) as surprising because the surprising theorem (33) is an immediate consequence of (32) and the apparently non-surprising theorem (25).

Mally's list of surprising theorems seems too short: for example, (24) is equivalent to $A \to !A$ in virtue of Def. f. But $A \to !A$ may be paraphrased as "the facts ought to be the case," an assertion which, as noted above, Mally did regard as surprising (Mally 1926, p. 24). So then why didn't he call (24) surprising? Did it not surprise him after (22)?

Even though Mally regarded many of his theorems as surprising, he thought that he had discovered an interesting concept of "correct willing" (*richtiges Wollen*) or "willing in accordance with the facts" which should not be confused with the notions of obligation and willing used in ordinary discourse. Mally's "exact system of pure ethics" was mainly concerned with this concept, but we will not describe this system because it belongs to the field of ethics rather than deontic logic.

6. Menger's Criticism

In 1939, Karl Menger published a devastating critique of Mally's formal system. He pointed out that the following formula is a theorem of this system:

$$\text{M.} \quad A \leftrightarrow !A$$

In other words, if A is the case then A ought to be the case, and if A ought to be the case then A is indeed the case. As we have already noted in connection with theorems (34) and (35), Mally made the same claim in informal terms, but the just-mentioned formula does not actually occur in his book.

Menger's theorem may be proven as follows. (Menger's own proof was a bit different.)

First, $A \to !A$ is a theorem (As noted above, this is equivalent to Mally's own (24).):

1.	$(A \to B) \to (!A \to !A)$	**PC**
2.	$(A \to B) \to (A \to B)$	**PC**
3.	$(A \to B) \to ((!A \to !A) \,\&\, (A \to B))$	1, 2
4.	$((!A \to !A) \,\&\, (A \to B)) \to (!A \to !B)$	I′
5.	$(A \to B) \to (!A \to !B)$	3, 4
6.	$!A \to !A$	**PC**
7.	$!(!A \to A)$	6, III′ →
8.	$(!A \to A) \to (A \to A)$	**PC**
9.	$!(!A \to A) \to !(A \to A)$	5, 8
10.	$!(A \to A)$	7, 9
11.	$A \to !A$	10, III′ ←

(Here line 9 follows from lines 5 and 8 first by substituting $!A \to A$ for A and $A \to A$ for B in 5 and then applying 8 with modus ponens. This derivation may seem longer than necessary; the reason why we give it in this form will be explained in the first paragraph of section 8 below.)

Second, $!A \to A$ is a theorem:

1.	$((U \to !A) \,\&\, (A \to \mho)) \to (U \to !\mho)$	I′
2.	$\neg((U \to !A) \,\&\, (A \to \mho))$	1, V′
3.	$\neg((U \to !A) \,\&\, (A \to \mho)) \to (!A \to A)$	**PC**
4.	$!A \to A$	2, 3

Because $A \to !A$ and $!A \to A$ are both theorems, $A \leftrightarrow !A$ is a theorem as well.

Menger gave the following comment:

> This result seems to me to be detrimental for Mally's theory, however. It indicates that the introduction of the sign ! is superfluous in the sense that it may be cancelled or inserted in any formula at any place we please. But this result (in spite of Mally's philosophical justification) clearly contradicts not only our use of the word "ought" but also some of Mally's own correct remarks about this concept, e.g. the one at the beginning of his development to the effect that $p \to (!q \text{ or } !r)$ and $p \to !(q \text{ or } r)$ are not equivalent. Mally is quite right that these two propositions are not equivalent according to the ordinary use of the word "ought." But they are equivalent according to his theory by virtue of the equivalence of p and $!p$ (Menger 1939, p. 58).

Almost all deontic logicians have accepted Menger's verdict. Indeed, it is generally considered a minimal condition of adequacy for a deontic logic that neither $A \to !A$ nor $!A \to A$ be derivable. Since 1939, Mally's deontic system has seldom been taken seriously.

7. Where Did Mally Go Wrong?

Where did Mally go wrong? How could one construct a system of deontic logic which does more justice to the notion of what ought to be the case that is used in ordinary discourse while still remaining as faithful as possible to Mally's basic intuitions? Three types of answers are possible:

- Mally should not have added his deontic axioms to classical propositional logic;
- Some of his deontic principles should be modified; and
- Both of the above. Menger advocated the latter kind of view: "One of the reasons for the failure of Mally's interesting attempt is that it was founded on the 2-valued calculus of propositions" (Menger 1939, p. 59).

We will only explore the first two suggestions. Each of them will turn out to be sufficient, so the third proposal is superfluous unless it is independently motivated.

We will first show that if Mally's deontic principles are added to a system in which the so-called paradoxes of material and strict implication are avoided, most of the "surprising" theorems, such as (34) and (35), are no longer derivable nor, most importantly, is $A \leftrightarrow !A$. But most of the theorems which Mally regarded as "plausible" remain. The resulting system is closely related to Anderson's relevant deontic logic (1967).

After this, we will show that one might also modify some of the specifically deontic principles, notably, Def. f and Axiom I. The resulting system is almost identical with the system nowadays known as standard deontic logic (**SDL**). Here too (34), (35) and $A \leftrightarrow !A$ are no longer derivable.

8. Alternative Non-Deontic Bases

When one looks at the derivations of many of Mally's surprising theorems, one finds they depend on such principles of classical logic as $A \rightarrow (B \rightarrow A)$, one of the paradoxes of material implication. This suggests that the implication and conditionals of Mally's informal postulates (i), (ii), (iii) and (v) should be formalized by a non-classical implication connective. Føllesdal and Hilpinen (1981, pp. 5–6) proposed that some sort of strict implication would be more appropriate. This proposal will not suffice, however, if by "strict implication" one means the connective of a typical modal logic. For, the proof of $A \rightarrow !A$ given above will still go through if Mally's axioms I′ and III′ are added to any modal logic that contains $(A \rightarrow B) \rightarrow (C \rightarrow C)$ (with $\rightarrow$ for strict implication). This includes all the normal modal logics, such as **K**, **T**, **S4** and **S5**, but also non-normal systems like **S2** and **S3**.

Although systems of strict implication do avoid the paradoxes of material implication, like $A \rightarrow (B \rightarrow A)$, they still contain other paradoxes, the paradoxes of strict implication, such as $(A \mathbin{\&} \neg A) \rightarrow B$ and $(A \rightarrow B) \rightarrow (C \rightarrow C)$, mentioned above. This leads us to consider what happens when Mally's axioms are added to a logical base that is paradox-free, i.e., to a system of relevant logic. Here we find that most of the theorems that Mally regarded as surprising are no longer derivable, nor, most importantly, are either $A \to\, !A$ or $!A \to A$, and yet many of his "plausible" theorems remain provable. Thus, on this approach, we suggest that Mally went wrong in basing his system on an underlying logic that contains fallacies of relevance (not that he had much choice when he was writing).

To see how this fault could be rectified, we add Mally's axioms to the strong relevant logic **R** (enriched with the propositional constant t, the "conjunction of all truths"). This has the following axioms and rules (Anderson & Belnap 1975, ch. V). $\leftrightarrow$ is defined by $A \leftrightarrow B = (A \to B) \,\&\, (B \to A)$:

Self-implication.	$A \to A$
Prefixing.	$(A \to B) \to ((C \to A) \to (C \to B))$
Contraction.	$(A \to (A \to B)) \to (A \to B)$
Permutation.	$(A \to (B \to C)) \to (B \to (A \to C))$
&Elimination.	$(A \,\&\, B) \to A$, $(A \,\&\, B) \to B$
&Introduction.	$((A \to B) \,\&\, (A \to C)) \to (A \to (B \,\&\, C))$
∨Introduction.	$A \to (A \vee B)$, $B \to (A \vee B)$
∨Elimination.	$((A \to C) \,\&\, (B \to C)) \to ((A \vee B) \to C)$
Distribution.	$(A \,\&\, (B \vee C)) \to ((A \,\&\, B) \vee C)$
Double Negation.	$\neg\neg A \to A$
Contraposition.	$(A \to \neg B) \to (B \to \neg A)$
Ax. t.	$A \leftrightarrow (t \to A)$
Modus Ponens.	$A, A \to B \;/\; B$
Adjunction.	$A, B \;/\; A \,\&\, B$

A relevant version **RD** (for "Relevant *Deontik*") of Mally's deontic system may be defined as follows:

- The language is the same as the language of **R**, except that we write V instead of t, add the propositional constant U and the unary connective !, and define $\mathit{\Lambda}$, $\mathit{\Omega}$, f and ∞ as in Mally's system.
- Axiomatization: add Mally's Axioms I–V to the axioms and rules of **R**.

It is, perhaps, worth noting that as we equate the Ackermann constant t of **R** with Mally's V we change the meaning of this term somewhat. For although $A \to (t \to A)$ is provable (part of axiom t) in **R**, $A \to t$ is not, while $A \to V$ is a fundamental theorem of classical logic when V represents an arbitrary tautology. The failure of $A \to t$, which is part of the spirit of relevant logics, is crucial to the results below.

As we compare **RD** with Mally's original system, we should note that strictly speaking his theorems (2), (5), (15) and (27) do not belong to the language of **RD** because they contain propositional quantifiers. If such quantifiers were added to **RD** in the usual way (Anderson, Belnap & Dunn 1992, sec. 32), then we could easily prove that these theorems are deductively equivalent to (2′), (5′), (15′) and (27′), respectively:

(2′)	$(A \mathsf{f} \Lambda) \to (A \mathsf{f} B)$
(5′)	$(!A \to (B \mathsf{f} A))\ \&\ ((V \mathsf{f} A) \to !A)$
(15′)	$A \mathsf{f} U$
(27′)	$\Omega \mathsf{f} A$

To avoid needless complications, we will not equip **RD** with quantifiers, but instead confine our attention to these unquantified versions of (2), (5), (15) and (27), none of which is provable in **RD**.

RD has the following properties.

- Axioms I, II and III may be replaced by the following three simpler axioms:

I*.	$(A \to B) \to (!A \to !B)$
II*.	$(!A\ \&\ !B) \to !(A\ \&\ B)$
III*.	$!(!A \to A)$

  The proof is as follows. First, I* is a theorem of **RD**:

1.	$(!A \to !A)\ \&\ (A \to ((A \to B) \to B))$	**R**
2.	$!A \to !((A \to B) \to B)$	1, Ax. I, Def. f
3.	$!A \to ((A \to B) \to !B)$	2, Ax. III←, Def. f
4.	$(A \to B) \to (!A \to !B)$	3, Permutation

  There are five other cases to be considered: II* and III* are theorems of **RD** and I–III follow from I*–III*. These cases are easy and left to the reader.[1]

- Formulas I′–V′, (3), (4), (6), (8)–(11), parts of (12) and (13), (14), (16)–(18), (23′) and (30) are theorems of **RD**.

  Proof: most cases are obvious but some hints for (16)–(18) and (30) may be helpful. (16) is a consequence of I* and IV. (17) follows from (16) and (18). (18) may be proven as follows:

1.	$!(!A \to A)\ \&\ !(!!A \to !A)$	III* (twice), Adj
2.	$!((!A \to A)\ \&\ (!!A \to !A))$	1, II*
3.	$((!A \to A)\ \&\ (!!A \to !A)) \to (!!A \to A)$	**R**
4.	$!((!A \to A)\ \&\ (!!A \to !A)) \to !(!!A \to A)$	3, I*
5.	$!(!!A \to A)$	2, 4
6.	$!!A \to !A$	5, Ax. III←, Def. f

  (30) is proven as follows: we have $\Omega \to (V \to \Omega)$ by Ax. t, whence $\Omega \to (U \to \Lambda)$ by Contraposition and Double Negation, whence $\Omega \to !\Lambda$ by (16).

[1] See p. 19 below.

- Formulas (1), (2), (5), (7), parts of (12) and (13), (15), (19)–(23), (24)–(29), (31)–(35), $A \to\, !A$ and $!A \to A$ are not derivable.

  This is demonstrated by the matrices given in the Appendix. (With (12) and (13) the only parts that fail are due to the non-equivalence of $A \to B$ and $\neg(A \mathbin{\&} \neg B)$ or $\neg A \vee B$; $(A \mathrel{f} B) \leftrightarrow (A \to\, !B) \leftrightarrow\, !(A \to B)$, $!\neg(A \mathbin{\&} \neg B) \leftrightarrow\, !(\neg A \vee B)$, and $\neg(A \mathbin{\&} \neg !B) \leftrightarrow (\neg A \vee\, !B)$ are provable in **RD**.)

- There are 12 mismatches between **RD** and Mally's expectations: (5), the noted parts of (12)–(13), (15), (19)–(21), (23) and (24)–(26) are not derivable even though Mally did not regard these formulas as surprising. Also, (30) is a theorem of **RD** even though Mally did consider it surprising.

- Formulas (34) and (35) (the formulas which Mally viewed as the most surprising theorems of his system) are in a sense stranger than Menger's theorem $A \leftrightarrow\, !A$ because the latter theorem is derivable in **RD** supplemented with (34) or (35) while neither (34) nor (35) is derivable in **RD** supplemented with $A \leftrightarrow\, !A$.

  The proof is as follows. First, Menger's theorem $A \leftrightarrow\, !A$ is a theorem of both **RD**+(34) and **RD**+(35):

1.	$A \leftrightarrow (V \to A)$	Ax. t
2.	$A \leftrightarrow (U \to A)$	1, either (34) or (35)
3.	$A \to\, !A$	2, (16)
4.	$(A \to \mho) \to ((U \to\, !A) \to (U \to\, !\mho))$	I*, Prefixing
5.	$(A \to \mho) \to (\neg(U \to\, !\mho) \to \neg(U \to\, !A))$	4, Contraposition
6.	$\neg(U \to\, !\mho) \to ((A \to \mho) \to \neg(U \to\, !A))$	5, Permutation
7.	$(A \to \mho) \to \neg(U \to\, !A)$	6, Ax. V, Def. f
8.	$!A \to A$	2, 7, **R**
9.	$A \leftrightarrow\, !A$	3, 8, Adjunction

  Second, neither (34) nor (35) is derivable in **RD** supplemented with $A \leftrightarrow\, !A$, as is shown in the Appendix.

Although most of Mally's surprising theorems are not derivable in **RD**, this has nothing to do with Mally's own reasons for regarding these theorems as surprising. They are not derivable in **RD** because they depend on fallacies of relevance. Mally never referred to such fallacies to explain his surprise. His considerations were quite different, as we have already described.

RD is closely related to relevant deontic logic **ARD** of Alan Ross Anderson (Anderson 1967, 1968, McArthur 1981). This is defined as **R** supplemented with the following two axioms, using the connective ! in place of Anderson's O:

$$\text{ARD1.} \quad !A \leftrightarrow (\neg A \to \mho)$$
$$\text{ARD2.} \quad !A \to \neg !\neg A$$

- All theorems of **RD** are theorems of **ARD**. The proof is easy.

- ARD1$\rightarrow$ is not a theorem of **RD**+ARD2: see the Appendix. This formula does not occur in Mally's book. (Anderson (1967, p. 348) credits Bohnert (1945) as the first to propose this equivalence, though he knew that its counterpart is found also in Menger (1939, p. 59).)

- ARD2 is not a theorem of **RD**+ARD1: see the Appendix. This formula does not occur in Mally's book, but Mally endorsed the corresponding informal principle: "a person who wills correctly does not will (not even implicitly) the negation of what he wills; correct willing is free of contradictions." (*Wer richtig will, will nicht (auch nicht impliziterweise) das Negat des Gewollten; richtiges Wollen ist widerspruchsfrei*—Mally 1926, p. 49.) Mally regarded this as a paraphrase of Axiom V, but this is not entirely correct. Morscher (1998, p. 106) has argued that ARD2 expresses Mally's intentions more adequately than Axiom V does.

- **RD** supplemented with ARD1$\rightarrow$ and ARD2 has the same theorems as **ARD**. Proof: ARD1 is a theorem of **RD**+ARD1$\rightarrow$ by virtue of theorem (16), i.e., ARD1$\leftarrow$. Note that Axiom V is redundant because we have $!U$ by Ax. IV, whence $\neg!\mho$ by ARD2, whence $\neg(U \rightarrow \mho)$ by ARD1, whence $\neg(U \rightarrow (U \rightarrow \mho))$ by **R**, whence $\neg(U \rightarrow !\mho)$ by ARD1, whence Axiom V by Def. f.

Anderson's system has several problematical features (McArthur 1981, Goble 1999, 2001) and **RD** shares most of these features. But we will not pursue those issues here. The important point, for our purposes, is that **RD** is superior to Mally's original system because $A \leftrightarrow !A$ is not forthcoming, while it still contains most of his plausible principles and only one that he found surprising.

9. Alternative Deontic Principles

From his introductory discussion of the non-deontic propositional logic he was using, it is apparent that Mally was committed to treating those connectives in a completely classical way. This raises the question of whether, instead of changing the non-deontic logical base of his system, one might modify the specifically deontic principles in a way that would both accord with his primary intentions and yet avoid the unacceptable consequences of his original system.

On this approach, we propose that Mally went wrong in two respects. First was his identification of "A requires B" with "if A then it ought to be that B", i.e., his Def. f. Second was the way he formalized his first informal principle (i) with Axiom I. This suggests the following revisions of Mally's principles. (Morscher (1998) has made a somewhat similar proposal, but the details are different.)

First, regard f as primitive, rather than defined in terms of $\rightarrow$ and ! by Def. f; this is in keeping with Mally's presentation of the axioms and his list of theorems that focus on the connective f. One might even consider Mally's *Deontik* as a "logic of requirement". With f as primitive, ! can then be defined by

$$\text{Def. !.} \quad !A = V \; f \; A$$

(Alternatively, both connectives can be treated as primitive and the effect of Def. ! achieved with an axiom, $!A \leftrightarrow V \; f \; A$, which is derivable in both Mally's original system and in **RD** and so should be unproblematic from either point of view.)

Second, replace Mally's Axiom I, which may also be written $(B \to C) \to ((A \; f \; B) \to (A \; f \; C))$, with its counterpart *rule of inference*:

$$\text{R}f. \quad B \to C \;/\; (A \; f \; B) \to (A \; f \; C)$$

This is suggested by Mally's statement of his informal principle (i), "If A requires B and B implies C, then A requires C"; one might suppose that in his formulation Mally confused material implication (if-then) with formal implication or entailment, which would be formalized by provable conditionals.

Let **MD** (for "Modified *Deontik*") be the result of adding Def. ! (or its axiomatic counterpart) and the rule Rf to **PC**, along with Axioms II, III, IV and V as originally stated. Then the following are derivable in **MD**:

1.	$B \to C \;/\; !B \to !C$	Def. !, Rf
2.	$(!A \,\&\, !B) \to !(A \,\&\, B)$	Def. !, Ax. II
3.	$!V$	1, Ax. IV, **PC**
4.	$\neg !\Lambda$	1, Ax. III←, Ax. V, **PC** (ex falso)

The so-called standard system of deontic logic **SDL**, or **KD**, is defined as **PC** supplemented with 1–4 (except that ! is usually written as O); V is an arbitrary tautology. Hence, **MD** is at least as strong as **SDL**. It is not difficult to see that it is in fact identical with **SDL** supplemented with OU (Mally's Axiom IV) and the following definition of f: $A \; f \; B = O(A \to B)$. In modern monadic deontic logic, the notion of *commitment* is sometimes defined in this way. (Compare Mally's Axiom III.) Furthermore, if U is identified with V or OV even the additional postulate OU is unnecessary.

In **MD**, Mally's theorems have the following status. (As before, we consider the unquantified versions of (2), (5), (15), and (27).)

- II′, IV′, (1)–(5), (7)–(11), most of (12), (13)–(15), (17), (20)–(24) and (27)–(32) are derivable. (The only part of (12) not derivable is the first equivalence $(A \, f \, B) \leftrightarrow (A \to !B)$, corresponding to Def. f; the other parts, $A \, f \, B \leftrightarrow !(A \to B) \leftrightarrow !\neg(A \,\&\, \neg B) \leftrightarrow !(\neg A \vee B)$, are all provable in **MD**.)
- I′, III′, V′, (6), the noted part of (12), (16), (18)–(19), (25)–(26), (33)–(35), $A \to !A$ and $!A \to A$ are not derivable.

- There are 20 mismatches with Mally's deontic expectations: 10 "plausible" formulas are no longer derivable, namely Def. f (or one part of (12)), I′, III′, V′, (6), (16), (18)–(19) and (25)–(26), and 10 "surprising" theorems are still derivable, namely (1), (2), (7), (22) and (27)–(32).
- Although (34) and (35) are not derivable, adding them would not lead to the theoremhood of $A \leftrightarrow\, !A$.

Since there were only 12 mismatches in the case of **RD**, perhaps **MD** does less justice to Mally's deontic expectations than **RD** did. But it agrees better with his general outlook because it is still based on classical propositional logic, a system with which he seemed quite comfortable.

Furthermore, although many of Mally's surprising theorems remain derivable in this modification of his system, they have, as it were, lost their sting. Those theorems lead to the particularly surprising consequences described earlier only when combined with Mally's original Axiom I and his definition of f. For example, although (1) continues to be a theorem, (1′), $(A\, f\, B) \to (C \to (A\, f\, C))$, which better expresses what Mally found surprising, is not provable. Without this, (1) itself seems harmless. The others are similar.

Because **MD** is equivalent to standard deontic logic, **SDL**, it faces the limitations of that system, including, for example, the various "paradoxes of deontic logic". As with **RD**, we need not pursue such issues here. Our purpose has been rather to show that this revised system is better than Mally's original proposal, because $A \leftrightarrow\, !A$ is not forthcoming, nor are the other genuinely surprising elements of his system.

10. Conclusion

Mally's original deontic logic is unacceptable for the reasons stated by Menger (1939). It contains $A \leftrightarrow\, !A$ and so reduces the notion of ought or obligation to truth. Because of this, later authors tend to dismiss Mally's system. They deny that it is a "real" deontic logic, and perhaps "mention it only by way of curiosity" (Meyer and Wieringa 1993, p. 4). A recent paper even asserts that "Mally's naivety is astonishing" (Weinberger 2001, p. 295).

We think Mally's system deserves better than this. It is not as bad as it might first appear, for only relatively minor modifications are needed to turn it into a more acceptable system. One may either change the non-deontic basis to obtain a system that is similar to Anderson's system, or apply two natural patches to the deontic postulates to end up with a system equivalent to standard deontic logic. Since it is only a small step, not a giant leap, from Mally's original deontic logic to modern systems, his pioneering effort merits rehabilitation rather than contempt.

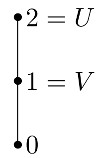
Appendix: Non-Derivability in RD and Some of Its Extensions

The following results were obtained with the help of MaGIC 2.1.4 (Slaney 1993).

First, consider the following 3-valued matrices and Hasse diagram:[2]

A	$\neg A$	$!A$
0	2	0
1	1	1
2	0	2

$A \to B$	B: 0	1	2
A: 0	2	2	2
1	0	1	2
2	0	0	2

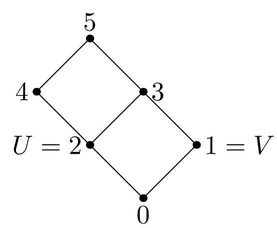


Each theorem A of **RD** + M + ARD2 has the following property: $v(A) \neq 0$ for each assignment v of the values 0, 1 and 2 to the variables in A. (We assume that $v(A \,\&\, B)$ is the greatest lower bound of $v(A)$ and $v(B)$ while $v(A \vee B)$ is the least upper bound of $v(A)$ and $v(B)$ as indicated in the diagram.) Formulas (1), (2), (5), (7), (12), (13), (21), (23), (31), (34), (35) and ARD1→ do not have this property.

Second, consider the following 6-valued matrices and Hasse diagram:

A	$\neg A$	$!A$
0	5	0
1	4	0
2	3	1
3	2	1
4	1	3
5	0	5

$A \to B$	B: 0	1	2	3	4	5
A: 0	5	5	5	5	5	5
1	0	1	2	3	4	5
2	0	0	1	1	3	5
3	0	0	0	1	2	5
4	0	0	0	0	1	5
5	0	0	0	0	0	5



Each theorem A of **RD** + ARD1→ has the following property: $v(A)$ is 1, 3 or 5 for each assignment v of the values 0, 1, 2, 3, 4 and 5 to the variables in A. Formulas (1), (2), (5), (7), (12), (13), (15), (19)–(23), (24)–(29), (31)–(35), ARD2, M→ and M← do not have this property.

Taken together, these results establish that (1), (2), (5), (7), (12), (13), (15), (19)–(23), (24)–(29), (31)–(35), ARD1→, ARD2, M→ and M← are not among the theorems of **RD**.

References

[1] Anderson, Alan Ross, 1967, "Some nasty problems in the formal logic of ethics," *Noûs*, vol. 1, pp. 345–360.

[2] Anderson, Alan Ross, 1968, "A new square of opposition: Eubouliatic logic," in *Akten des XIV. Internationalen Kongresses für Philosophie*, vol. 2, pp. 271–284, Vienna: Herder.

[2] See p. 19 below.

[3] Anderson, Alan Ross, and Nuel D. Belnap, Jr., 1975, *Entailment: The Logic of Relevance and Necessity*, vol. 1, Princeton, N. J.: Princeton University Press.

[4] Anderson, Alan Ross, Nuel D. Belnap, Jr., and J. Michael Dunn, 1992, *Entailment: The Logic of Relevance and Necessity*, vol. 2, Princeton, N. J.: Princeton University Press.

[5] Bohnert, Herbert G., 1945, "The semiotic status of commands," *Philosophy of Science*, vol. 12, pp. 302–315.

[6] Føllesdal, Dagfinn, and Risto Hilpinen, 1981, "Deontic logic: An introduction," in Risto Hilpinen, ed., *Deontic Logic: Introductory and Systematic Readings*, pp. 1–35, Dordrecht: D. Reidel, 2nd edn.

[7] Goble, Lou, 1999, "Deontic logic with relevance," in Paul McNamara and Henry Prakken, eds., *Norms, Logics and Information Systems: New Studies in Deontic Logic and Computer Science*, pp. 331–345, Amsterdam, etc.: IOS Press.

[8] Goble, Lou, 2001, "The Andersonian reduction and relevant deontic logic," in Bryson Brown and John Woods, eds., *New Studies in Exact Philosophy: Logic, Mathematics and Science—Proceedings of the 1999 Conference of the Society of Exact Philosophy*, pp. 213–246, Paris: Hermes Science Publications.

[9] Lokhorst, Gert-Jan C., 1999, "Ernst Mally's *Deontik* (1926)," *Notre Dame Journal of Formal Logic*, vol. 40, pp. 273–282.

[10] Mally, Ernst, 1926, *Grundgesetze des Sollens: Elemente der Logik des Willens*. Graz: Leuschner und Lubensky, Universitäts-Buchhandlung, viii+85 pp. Reprinted in Ernst Mally, *Logische Schriften: Großes Logikfragment, Grundgesetze des Sollens*, edited by Karl Wolf and Paul Weingartner, pp. 227–324, Dordrecht: D. Reidel, 1971.

[11] McArthur, Robert P., 1981, "Anderson's deontic logic and relevant implication," *Notre Dame Journal of Formal Logic*, vol. 22, pp. 145–154.

[12] Menger, Karl, 1939, "A logic of the doubtful: On optative and imperative logic," in *Reports of a Mathematical Colloquium*, 2nd series, 2nd issue, pp. 53–64, Notre Dame, Indiana: Indiana University Press.

[13] Meyer, John-Jules Ch., and Roel J. Wieringa, 1993, "Deontic logic: A concise overview," in John-Jules Ch. Meyer and Roel J. Wieringa, eds., *Deontic Logic in Computer Science*, pp. 3–16, Chichester: John Wiley and Sons.

[14] Morscher, Edgar, 1998, "Mallys Axiomsystem für die deontische Logik: Rekonstruktion und kritische Würdigung," in Alexander Hieke, ed., *Ernst Mally: Versuch einer Neubewertung*, pp. 81–165, Sankt Augustin: Academia Verlag.

[15] Slaney, John, 1993, *MaGIC: Matrix Generator for Implication Connectives*. Canberra, Australia: Automated Reasoning Project, Research School of Information Sciences and Engineering, Australian National University. Available at `ftp://csl.anu.edu.au/pub/MaGIC/magic.2.1.4.tar.gz`.

[16] Weinberger, Ota, 2001, "Ernst Mallys Deontik: Ein kritischer Rückblick und ein konstruktiver Ausblick nach einem dreiviertel Jahrhundert," in Thomas Binder, Reinhard Fabian, Ulf Höfer and Jutta Valent, eds., *Bausteine zu einer Geschichte der Philosophie an der Universität Graz*, pp. 289–303, Amsterdam/Atlanta: Rodopi.

[17] Whitehead, Alfred North, and Bertrand Russell, 1910, *Principia Mathematica*, vol. 1, Cambridge: Cambridge University Press.

[18] Jan Woleński, 1998, "Remarks on Mally's *Deontik*," in Alexander Hieke, ed., *Ernst Mally: Versuch einer Neubewertung*, pp. 73–80, Sankt Augustin: Academia Verlag.

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Addenda et Corrigenda

Note to p. 11

Two additional hints may be helpful.

First, axiom III$\leftarrow$ is derivable from I*:

1.	$(A \to B) \to (A \to B)$	Self-implication
2.	$A \to ((A \to B) \to B)$	1, Permutation
3.	$((A \to B) \to B) \to (!(A \to B) \to !B)$	I*
4.	$A \to (!(A \to B) \to !B)$	2, 3
5.	$!(A \to B) \to (A \to !B)$	4, Permutation

Second, axiom III$\to$ is derivable from I* and III* (and III$\leftarrow$, which has already been shown to be derivable from I*):

1.	$(!B \to B) \to ((A \to !B) \to (A \to B))$	Prefixing
2.	$!(!B \to B) \to !((A \to !B) \to (A \to B))$	1, I*
3.	$!((A \to !B) \to (A \to B))$	2, III*
4.	$(A \to !B) \to !(A \to B)$	3, III$\leftarrow$

Note to p. 16

The first column of the first matrix is incorrect in the paper printed in the *Grazer philosophische Studien*.

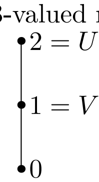
Longer version of the appendix (not in original paper)

Observation A Consider the following 3-valued matrices:

$\neg$	
0	2
1	1
2	0

!	
0	0
1	1
2	2

$\to$	0	1	2
0	2	2	2
1	0	1	2
2	0	0	2



Each theorem A of **RD** + M + ARD2 has the following property: $1 \sqsubseteq v(A)$ for each assignment v of the values 0, 1, and 2 to the variables in A. Formulas (1)–(2), (5), (7), (12c), (13a), (21), (23), (31), (34)–(35), and ARD1$\to$ do not have this property, so they are not among the theorems of **RD** + M + ARD2.

Self-impl	$A \to A$	$\surd$
Pref	$(A \to B) \to ((C \to A) \to (C \to B))$	$\surd$
Contract	$(A \to (A \to B)) \to (A \to B)$	$\surd$
Perm	$(A \to (B \to C)) \to (B \to (A \to C))$	$\surd$
DblNeg	$\neg\neg A \to A$	$\surd$
Contrapos	$(A \to \neg B) \to (B \to \neg A)$	$\surd$
&Elim1	$(A \,\&\, B) \to A$	$\surd$
&Elim2	$(A \,\&\, B) \to B$	$\surd$
&Int	$((A \to B) \,\&\, (A \to C)) \to (A \to (B \,\&\, C))$	$\surd$
∨Int1	$A \to (A \vee B)$	$\surd$
∨Int2	$B \to (A \vee B)$	$\surd$

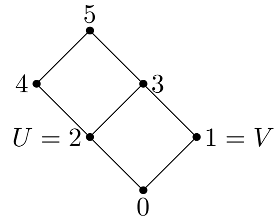
∨Elim	$((A \to C) \,\&\, (B \to C)) \to ((A \lor B) \to C)$	$\surd$
Distr	$(A \,\&\, (B \lor C)) \to ((A \,\&\, B) \lor C)$	$\surd$
Ax. V	$A \leftrightarrow (V \to A)$	$\surd$
MP	$A, (A \to B)/B$	$\surd$
Adj	$A, B/(A \,\&\, B)$	$\surd$
I	$((A\, f\, B) \,\&\, (B \to C)) \to (A\, f\, C)$	$\surd$
II	$((A\, f\, B) \,\&\, (A\, f\, C)) \to (A\, f\, (B \,\&\, C))$	$\surd$
III	$(A\, f\, B) \leftrightarrow !(A \to B)$	$\surd$
IV	$!U$	$\surd$
V	$\neg(U\, f\, \Omega)$	$\surd$
(1)	$(A\, f\, B) \to (A\, f\, V)$	$(1\, f\, 2) \to (1\, f\, 1) = 0$
(2)	$(A\, f\, \Lambda) \to (A\, f\, B)$	$(1\, f\, 1) \to (1\, f\, 0) = 0$
(3)	$((A\, f\, B) \lor (A\, f\, C)) \to (A\, f\, (B \lor C))$	$\surd$
(4)	$((A\, f\, B) \,\&\, (C\, f\, D)) \to ((A \,\&\, C)\, f\, (B \,\&\, D))$	$\surd$
(5)	$(!A \to (B\, f\, A)) \,\&\, ((V\, f\, A) \to !A)$	$(!1 \to (2\, f\, 1)) \,\&\, ((1\, f\, 1) \to !1) = 0$
(6)	$(!A \,\&\, (A \to B))\, f\, B$	$\surd$
(7)	$!A \to !V$	$!2 \to !1 = 0$
(8)	$((A\, f\, B) \,\&\, (B\, f\, C)) \to (A\, f\, C)$	$\surd$
(9)	$(!A \,\&\, (A\, f\, B))\, f\, B$	$\surd$
(10)	$(!A \,\&\, !B) \leftrightarrow !(A \,\&\, B)$	$\surd$
(11)	$(A \infty B) \leftrightarrow !(A \leftrightarrow B)$	$\surd$
(12a)	$(A\, f\, B) \leftrightarrow (A \to !B)$	$\surd$
(12b)	$(A \to !B) \leftrightarrow !(A \to B)$	$\surd$
(12c)	$!(A \to B) \leftrightarrow !\neg(A \,\&\, \neg B)$	$!(1 \to 0) \leftrightarrow !\neg(1 \,\&\, \neg 0) = 0$
(12d)	$!\neg(A \,\&\, \neg B) \leftrightarrow !(\neg A \lor B)$	$\surd$
(13a)	$(A\, f\, B) \leftrightarrow \neg(A \,\&\, \neg !B)$	$(1\, f\, 0) \leftrightarrow \neg(1 \,\&\, \neg !0) = 0$
(13b)	$\neg(A \,\&\, \neg !B) \leftrightarrow (\neg A \lor !B)$	$\surd$
(14)	$(A\, f\, B) \leftrightarrow (\neg B\, f\, \neg A)$	$\surd$
(15)	$A\, f\, U$	$\surd$
(16)	$(U \to A)\, f\, A$	$\surd$
(17)	$(U\, f\, A)\, f\, A$	$\surd$
(18)	$!!A \to !A$	$\surd$
(19)	$!!A \leftrightarrow !A$	$\surd$
(20)	$(U\, f\, A) \leftrightarrow (A \infty U)$	$\surd$
(21)	$!A \leftrightarrow (A \infty U)$	$!1 \leftrightarrow (1 \infty 2) = 0$
(22)	$!V$	$\surd$
(23)	$V \infty U$	$1 \infty 2 = 0$
(23′)	$V\, f\, U$	$\surd$
(24)	$A\, f\, A$	$\surd$
(25)	$(A \to B) \to (A\, f\, B)$	$\surd$
(26)	$(A \leftrightarrow B) \to (A \infty B)$	$\surd$
(27)	$\Omega\, f\, A$	$\surd$
(28)	$\Omega\, f\, \Omega$	$\surd$
(29)	$\Omega\, f\, U$	$\surd$
(30)	$\Omega\, f\, \Lambda$	$\surd$
(31)	$\Omega \infty \Lambda$	$0 \infty 1 = 0$
(32)	$\neg(U\, f\, \Lambda)$	$\surd$
(33)	$\neg(U \to \Lambda)$	$\surd$
(34)	$U \leftrightarrow V$	$2 \leftrightarrow 1 = 0$
(35)	$\Omega \leftrightarrow \Lambda$	$0 \leftrightarrow 1 = 0$
ARD1←	$(\neg A \to \Omega) \to !A$	$\surd$
ARD1→	$!A \to (\neg A \to \Omega)$	$!1 \to (\neg 1 \to 0) = 0$
ARD2	$!A \to \neg!\neg A$	$\surd$
M→	$A \to !A$	$\surd$
M←	$!A \to A$	$\surd$

Observation B Consider the following 6-valued matrices:

$\neg$	
0	5
1	4
2	3
3	2
4	1
5	0

$!$	
0	0
1	0
2	1
3	1
4	3
5	5

$\to$	0	1	2	3	4	5
0	5	5	5	5	5	5
1	0	1	2	3	4	5
2	0	0	1	1	3	5
3	0	0	0	1	2	5
4	0	0	0	0	1	5
5	0	0	0	0	0	5



Each theorem A of **RD** + ARD1$\to$ has the following property: $1 \sqsubseteq v(A)$ for each assignment v of the values 0, 1, 2, 3, 4, and 5 to the variables in A. Formulas (1)–(2), (5), (7), (12c), (13a), (15), (19)–(23), (24)–(29), (31)–(35), ARD2, M$\to$, and M$\leftarrow$ do not have this property, so they are not among the theorems of **RD** + ARD1$\to$.

Self-impl	$A \to A$	$\surd$
Pref	$(A \to B) \to ((C \to A) \to (C \to B))$	$\surd$
Contract	$(A \to (A \to B)) \to (A \to B)$	$\surd$
Perm	$(A \to (B \to C)) \to (B \to (A \to C))$	$\surd$
DblNeg	$\neg\neg A \to A$	$\surd$
Contrapos	$(A \to \neg B) \to (B \to \neg A)$	$\surd$
&Elim1	$(A \,\&\, B) \to A$	$\surd$
&Elim2	$(A \,\&\, B) \to B$	$\surd$
&Int	$((A \to B) \,\&\, (A \to C)) \to (A \to (B \,\&\, C))$	$\surd$
$\vee$Int1	$A \to (A \vee B)$	$\surd$
$\vee$Int2	$B \to (A \vee B)$	$\surd$
$\vee$Elim	$((A \to C) \,\&\, (B \to C)) \to ((A \vee B) \to C)$	$\surd$
Distr	$(A \,\&\, (B \vee C)) \to ((A \,\&\, B) \vee C)$	$\surd$
Ax. V	$A \leftrightarrow (V \to A)$	$\surd$
MP	$A, (A \to B)/B$	$\surd$
Adj	$A, B/(A \,\&\, B)$	$\surd$
I	$((A \mathbin{f} B) \,\&\, (B \to C)) \to (A \mathbin{f} C)$	$\surd$
II	$((A \mathbin{f} B) \,\&\, (A \mathbin{f} C)) \to (A \mathbin{f} (B \,\&\, C))$	$\surd$
III	$(A \mathbin{f} B) \leftrightarrow {!}(A \to B)$	$\surd$
IV	$!U$	$\surd$
V	$\neg(U \mathbin{f} \Omega)$	$\surd$
(1)	$(A \mathbin{f} B) \to (A \mathbin{f} V)$	$(1 \mathbin{f} 2) \to (1 \mathbin{f} 1) = 0$
(2)	$(A \mathbin{f} \Lambda) \to (A \mathbin{f} B)$	$(1 \mathbin{f} 4) \to (1 \mathbin{f} 0) = 0$
(3)	$((A \mathbin{f} B) \vee (A \mathbin{f} C)) \to (A \mathbin{f} (B \vee C))$	$\surd$
(4)	$((A \mathbin{f} B) \,\&\, (C \mathbin{f} D)) \to ((A \,\&\, C) \mathbin{f} (B \,\&\, D))$	$\surd$
(5)	$(!A \to (B \mathbin{f} A)) \,\&\, ((V \mathbin{f} A) \to {!}A)$	$(!2 \to (2 \mathbin{f} 2)) \,\&\, ((1 \mathbin{f} 2) \to {!}2) = 0$
(6)	$(!A \,\&\, (A \to B)) \mathbin{f} B$	$\surd$
(7)	$!A \to {!}V$	$!2 \to {!}1 = 0$
(8)	$((A \mathbin{f} B) \,\&\, (B \mathbin{f} C)) \to (A \mathbin{f} C)$	$\surd$
(9)	$(!A \,\&\, (A \mathbin{f} B)) \mathbin{f} B$	$\surd$
(10)	$(!A \,\&\, {!}B) \leftrightarrow {!}(A \,\&\, B)$	$\surd$
(11)	$(A \infty B) \leftrightarrow {!}(A \leftrightarrow B)$	$\surd$
(12a)	$(A \mathbin{f} B) \leftrightarrow (A \to {!}B)$	$\surd$
(12b)	$(A \to {!}B) \leftrightarrow {!}(A \to B)$	$\surd$
(12c)	$!(A \to B) \leftrightarrow {!}\neg(A \,\&\, \neg B)$	$!(1 \to 0) \leftrightarrow {!}\neg(1 \,\&\, \neg 0) = 0$
(12d)	$!\neg(A \,\&\, \neg B) \leftrightarrow {!}(\neg A \vee B)$	$\surd$
(13a)	$(A \mathbin{f} B) \leftrightarrow \neg(A \,\&\, \neg{!}B)$	$(1 \mathbin{f} 0) \leftrightarrow \neg(1 \,\&\, \neg{!}0) = 0$
(13b)	$\neg(A \,\&\, \neg{!}B) \leftrightarrow (\neg A \vee {!}B)$	$\surd$
(14)	$(A \mathbin{f} B) \leftrightarrow (\neg B \mathbin{f} \neg A)$	$\surd$
(15)	$A \mathbin{f} U$	$2 \mathbin{f} 2 = 0$
(16)	$(U \to A) \mathbin{f} A$	$\surd$
(17)	$(U \mathbin{f} A) \mathbin{f} A$	$\surd$

(18)	$!!A \to !A$	$\surd$
(19)	$!!A \leftrightarrow !A$	$!!2 \leftrightarrow !2 = 0$
(20)	$(U\ f\ A) \leftrightarrow (A \infty U)$	$(2\ f\ 4) \leftrightarrow (4 \infty 2) = 0$
(21)	$!A \leftrightarrow (A \infty U)$	$!2 \leftrightarrow (2 \infty 2) = 0$
(22)	$!V$	$!1 = 0$
(23)	$V \infty U$	$1 \infty 2 = 0$
(23′)	$V\ f\ U$	$\surd$
(24)	$A\ f\ A$	$1\ f\ 1 = 0$
(25)	$(A \to B) \to (A\ f\ B)$	$(1 \to 1) \to (1\ f\ 1) = 0$
(26)	$(A \leftrightarrow B) \to (A \infty B)$	$(1 \leftrightarrow 1) \to (1 \infty 1) = 0$
(27)	$\Omega\ f\ A$	$3\ f\ 0 = 0$
(28)	$\Omega\ f\ \Omega$	$3\ f\ 3 = 0$
(29)	$\Omega\ f\ U$	$3\ f\ 2 = 0$
(30)	$\Omega\ f\ \Lambda$	$\surd$
(31)	$\Omega \infty \Lambda$	$3 \infty 4 = 0$
(32)	$\neg(U\ f\ \Lambda)$	$\neg(2\ f\ 4) = 4$
(33)	$\neg(U \to \Lambda)$	$\neg(2 \to 4) = 2$
(34)	$U \leftrightarrow V$	$2 \leftrightarrow 1 = 0$
(35)	$\Omega \leftrightarrow \Lambda$	$3 \leftrightarrow 4 = 0$
ARD1←	$(\neg A \to \Omega) \to !A$	$\surd$
ARD1→	$!A \to (\neg A \to \Omega)$	$\surd$
ARD2	$!A \to \neg!\neg A$	$!2 \to \neg!\neg 2 = 4$
M→	$A \to !A$	$1 \to !1 = 0$
M←	$!A \to A$	$!2 \to 2 = 2$

Observation C (from A and B) Formulas (1)–(2), (5), (7), (12c), (13a), (15), (19)–(23), (24)–(29), (31)–(35), ARD1→, ARD2, M→, and M← are not among the theorems of **RD**.

Afterthought regarding "the unconditionally obligatory"

We define $\mathbf{RD}^+$ as follows.

Language: **RD** without U. Definition: $U_A = !A \to A$.

Axioms and rules: **RD** without axioms IV–V supplemented with axiom VI: $\neg(U_A \to !\neg U_A)$.

Observations:

1. Axiom VI of $\mathbf{RD}^+$ is not a theorem of **RD** (proof: by MaGIC).
2. Although $!U$, $\neg(U \to !\neg U)$, and $!A \leftrightarrow (U \to A)$ do not belong to the language of $\mathbf{RD}^+$, $\mathbf{RD}^+$ does have the following theorems: $!U_A$, $\neg(U_A \to !\neg U_A)$, and $!A \leftrightarrow (U_A \to A)$.
3. Axiom ARD2 of **ARD** is not a theorem of $\mathbf{RD}^+$ (proof: by MaGIC).

Provided that we confine our attention to the formulas of **RD** and **ARD** that belong to the language of $\mathbf{RD}^+$, Theorems(**RD**) $\subset$ Theorems($\mathbf{RD}^+$) $\subset$ Theorems(**ARD**).

We view $\mathbf{RD}^+$ as an attractive U-free compromise between **RD** and **ARD**.